An optimal order finite element method for elliptic interface problems

Gunther Peichl
University of Graz, Austria

Rachid Touzani
Université Blaise Pascal (Clermont-Ferrand, France)

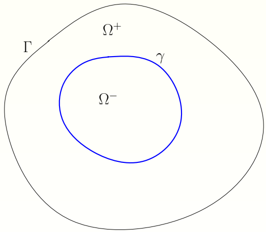
Consider the elliptic problem

$$\begin{aligned} -\nabla \cdot (a \nabla u) &= f && \text{in } \Omega \subset \mathbb{R}^2 \\ u &= 0 && \text{on } \Gamma := \partial\Omega \end{aligned}$$

where $f \in L^2(\Omega)$ and

$$\Omega = \Omega^- \cup \gamma \cup \Omega^+.$$

Here γ is a closed curve:



$a \in W^{1,\infty}(\Omega^+) \cap W^{1,\infty}(\Omega^-)$ and a is discontinuous across γ.

It is well known that, if the curves Γ and γ are “regular”, then we have

$$u \in H^2(\Omega^-) \cap H^2(\Omega^+), \text{ but } u \notin H^2(\Omega).$$

Remark

Even if γ is polygonal, then $u \notin H^2(\Omega^-) \cap H^2(\Omega^+)$. We have in general $u \in H^{\frac{3}{2}-\theta}$ for $\theta > 0$.

This model problem exhibits the same type of singularity as interface problems involved in:

- Free boundary problems
- Transmission problems
- Fictitious domain methods
- ...

Aim

To construct an accurate finite element method that does not fit the mesh.

It is well known that, if the curves Γ and γ are "regular", then we have

$$u \in H^2(\Omega^-) \cap H^2(\Omega^+), \text{ but } u \notin H^2(\Omega).$$

Remark

Even if γ is polygonal, then $u \notin H^2(\Omega^-) \cap H^2(\Omega^+)$. We have in general $u \in H^{\frac{3}{2}-\theta}$ for $\theta > 0$.

This model problem exhibits the same type of singularity as interface problems involved in:

- Free boundary problems
- Transmission problems
- Fictitious domain methods
- ...

Aim

To construct an accurate finite element method that does not fit the mesh.

It is well known that, if the curves Γ and γ are "regular", then we have

$$u \in H^2(\Omega^-) \cap H^2(\Omega^+), \text{ but } u \notin H^2(\Omega).$$

Remark

Even if γ is polygonal, then $u \notin H^2(\Omega^-) \cap H^2(\Omega^+)$. We have in general $u \in H^{\frac{3}{2}-\theta}$ for $\theta > 0$.

This model problem exhibits the same type of singularity as interface problems involved in:

- Free boundary problems
- Transmission problems
- Fictitious domain methods
- . . .

Aim

To construct an accurate finite element method that does not fit the mesh.

It is well known that, if the curves Γ and γ are "regular", then we have

$$u \in H^2(\Omega^-) \cap H^2(\Omega^+), \text{ but } u \notin H^2(\Omega).$$

Remark

Even if γ is polygonal, then $u \notin H^2(\Omega^-) \cap H^2(\Omega^+)$. We have in general $u \in H^{\frac{3}{2}-\theta}$ for $\theta > 0$.

This model problem exhibits the same type of singularity as interface problems involved in:

- Free boundary problems
- Transmission problems
- Fictitious domain methods
- ...

Aim

To construct an accurate finite element method that does not fit the mesh.

Some works related to this topic:

- Belytschko, Moës *et al.*: XFEM (eXtended Finite Element Method)
- Lamichhane and Wohlmuth: Mortar finite elements for interface problems
- Hansbo *et al.*: An unfitted finite element method . . .
- Z. Li: Immersed boundary techniques for interface problems
- . . .

Some works related to this topic:

- Belytschko, Moës *et al.*: XFEM (eXtended Finite Element Method)
- Lamichhane and Wohlmuth: Mortar finite elements for interface problems
- Hansbo *et al.*: An unfitted finite element method . . .
- Z. Li: Immersed boundary techniques for interface problems
- . . .

Some works related to this topic:

- Belytschko, Moës *et al.*: XFEM (eXtended Finite Element Method)
- Lamichhane and Wohlmuth: Mortar finite elements for interface problems
- Hansbo *et al.*: An unfitted finite element method ...
- Z. Li: Immersed boundary techniques for interface problems
- ...

Some works related to this topic:

- Belytschko, Moës *et al.*: XFEM (eXtended Finite Element Method)
- Lamichhane and Wohlmuth: Mortar finite elements for interface problems
- Hansbo *et al.*: An unfitted finite element method ...
- Z. Li: Immersed boundary techniques for interface problems
- ...

Some works related to this topic:

- Belytschko, Moës *et al.*: XFEM (eXtended Finite Element Method)
- Lamichhane and Wohlmuth: Mortar finite elements for interface problems
- Hansbo *et al.*: An unfitted finite element method ...
- Z. Li: Immersed boundary techniques for interface problems
- ...

A fitted finite element method

Assume that Ω is polygonal and consider a finite element mesh $\mathscr{T}_h$ of $\overline{\Omega}$. The simplest finite element method is given by the space

$$V_h = \{v \in C^0(\overline{\Omega});\ v_{|T} \in P_1(T)\ \forall\ T \in \mathscr{T}_h,\ v = 0 \text{ on } \Gamma\}.$$

The discrete problem is given by

$$\int_\Omega a \nabla u_h \cdot \nabla v \, dx = \int_\Omega fv \, dx \qquad \forall\ v \in V_h.$$

Classic error estimates

$$\|u - u_h\|_{1,\Omega} \le Ch$$

do not hold any more.

To construct a fitted FEM, we consider:

1. A piecewise linear approximation γ_h of the curve γ that implies a subdivision

$$\Omega = \Omega_h^- \cup \gamma_h \cup \Omega_h^+.$$

2. A subdivision of any "interface triangle" into 3 (or 2 in some cases) triangles

Assume that Ω is polygonal and consider a finite element mesh $\mathscr{T}_h$ of $\overline{\Omega}$. The simplest finite element method is given by the space

$$V_h = \{v \in C^0(\overline{\Omega});\ v_{|T} \in P_1(T)\ \forall\ T \in \mathscr{T}_h,\ v = 0 \text{ on } \Gamma\}.$$

The discrete problem is given by

$$\int_\Omega a \nabla u_h \cdot \nabla v \, dx = \int_\Omega fv \, dx \qquad \forall\ v \in V_h.$$

Classic error estimates

$$\|u - u_h\|_{1,\Omega} \leq Ch$$

do not hold any more.

To construct a fitted FEM, we consider:

1. A piecewise linear approximation γ_h of the curve γ that implies a subdivision

$$\Omega = \Omega_h^- \cup \gamma_h \cup \Omega_h^+.$$

2. A subdivision of any "interface triangle" into 3 (or 2 in some cases) triangles

A fitted finite element method

Assume that Ω is polygonal and consider a finite element mesh $\mathscr{T}_h$ of $\overline{\Omega}$. The simplest finite element method is given by the space

$$V_h = \{v \in C^0(\overline{\Omega});\ v_{|T} \in P_1(T)\ \forall\ T \in \mathscr{T}_h,\ v = 0 \text{ on } \Gamma\}.$$

The discrete problem is given by

$$\int_\Omega a\,\nabla u_h \cdot \nabla v\,dx = \int_\Omega fv\,dx \qquad \forall\ v \in V_h.$$

Classic error estimates

$$\|u - u_h\|_{1,\Omega} \le Ch$$

do not hold any more.

To construct a fitted FEM, we consider:

1. A piecewise linear approximation γ_h of the curve γ that implies a subdivision

$$\Omega = \Omega_h^- \cup \gamma_h \cup \Omega_h^+.$$

2. A subdivision of any "interface triangle" into 3 (or 2 in some cases) triangles

A fitted finite element method

Assume that Ω is polygonal and consider a finite element mesh $\mathscr{T}_h$ of $\overline{\Omega}$. The simplest finite element method is given by the space

$$V_h = \{v \in C^0(\overline{\Omega});\ v_{|T} \in P_1(T)\ \forall\ T \in \mathscr{T}_h,\ v = 0 \text{ on } \Gamma\}.$$

The discrete problem is given by

$$\int_\Omega a \nabla u_h \cdot \nabla v \, dx = \int_\Omega fv \, dx \qquad \forall\ v \in V_h.$$

Classic error estimates

$$\|u - u_h\|_{1,\Omega} \leq Ch$$

do not hold any more.

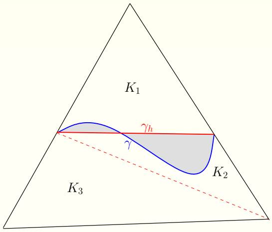
To construct a fitted FEM, we consider:

1. A piecewise linear approximation γ_h of the curve γ that implies a subdivision

$$\Omega = \Omega_h^- \cup \gamma_h \cup \Omega_h^+.$$

2. A subdivision of any "interface triangle" into 3 (or 2 in some cases) triangles

A fitted finite element method (Cont'd)



Subdivision of an interface triangle

Notations:

$\mathscr{T}_h^\gamma := \{T \in \mathscr{T}_h;\ \gamma \cap T^\circ \neq \emptyset\}$	Interface triangles
$\mathscr{E}_h^\gamma := \{e \text{ edge};\ \gamma \cap e^\circ \neq \emptyset\}$	Edges intersected by γ(or γ_h)
$\mathscr{T}_T^\gamma := \cup\{\text{subtriangles of } T\}$	
$\mathscr{T}_h^F := \mathscr{T}_h \cup \bigcup_{T \in \mathscr{T}_h^\gamma} \left(\cup_{K \in \mathscr{T}_T^\gamma} K \right)$	New fitted mesh
$S_h^\gamma := \bigcup \{T;\ T \in \mathscr{T}_h^\gamma\}$	Layer containing the interface

We next define an extension $\widetilde{a}_h$ of a and a piecewise linear interpolant a_h of $\widetilde{a}_h$, with

$$a_{h|\Omega_h^-} \in W^{1,\infty}(\Omega_h^-),\ a_{h|\Omega_h^+} \in W^{1,\infty}(\Omega_h^+),$$

a_h is discontinuous across γ_h,

$$\|a_h\|_{0,\infty,\Omega} \le C\,\|a\|_{0,\infty,\Omega}.$$

The fitted finite element space is given by

$$W_h := V_h + X_h\ (\subset H_0^1(\Omega))$$
$$X_h := \{v \in C^0(\overline{\Omega});\ v_{|\Omega\setminus S_h^\gamma} = 0,\ v_{|K} \in P_1(K)\ \forall\, K \in \mathscr{T}_T^\gamma,\ \forall\, T \in \mathcal{T}_h^\gamma\}$$

Whence the Fitted Finite Element Method:

Find $u_h^F \in W_h$ such that $\displaystyle\int_\Omega a_h \nabla u_h^F \cdot \nabla v\,dx = \int_\Omega fv\,dx \qquad \forall\, v \in W_h.$

We assume (a weaker mesh regularity) that for some $\theta \in [0,1)$,

$$\frac{h}{\varrho_K} \le C\,h^{-\theta} \qquad \forall\, K \in \mathscr{T}_T^\gamma,\ T \in \mathscr{T}_h^\gamma.$$

where ϱ_K is the diameter of the inscribed circle in K.

We next define an extension $\widetilde{a}_h$ of a and a piecewise linear interpolant a_h of $\widetilde{a}_h$, with

$$a_{h|\Omega_h^-} \in W^{1,\infty}(\Omega_h^-),\; a_{h|\Omega_h^+} \in W^{1,\infty}(\Omega_h^+),$$
$$a_h \text{ is discontinuous across } \gamma_h,$$
$$\|a_h\|_{0,\infty,\Omega} \leq C\,\|a\|_{0,\infty,\Omega}.$$

The fitted finite element space is given by

$$W_h := V_h + X_h \;(\subset H_0^1(\Omega))$$
$$X_h := \{v \in C^0(\overline{\Omega});\; v_{|\Omega \setminus S_h^\gamma} = 0,\; v_{|K} \in P_1(K)\;\forall\, K \in \mathscr{T}_T^\gamma,\;\forall\, T \in \mathcal{T}_h^\gamma\}$$

Whence the Fitted Finite Element Method:

Find $u_h^F \in W_h$ such that $$\int_\Omega a_h \nabla u_h^F \cdot \nabla v\, dx = \int_\Omega fv\, dx \qquad \forall\, v \in W_h.$$

We assume (a weaker mesh regularity) that for some $\theta \in [0,1)$,

$$\frac{h}{\varrho_K} \leq C\, h^{-\theta} \qquad \forall\, K \in \mathscr{T}_T^\gamma,\; T \in \mathscr{T}_h^\gamma.$$

where ϱ_K is the diameter of the inscribed circle in K.

We next define an extension $\widetilde{a}_h$ of a and a piecewise linear interpolant a_h of $\widetilde{a}_h$, with

$$a_{h|\Omega_h^-} \in W^{1,\infty}(\Omega_h^-),\; a_{h|\Omega_h^+} \in W^{1,\infty}(\Omega_h^+),$$
$$a_h \text{ is discontinuous across } \gamma_h,$$
$$\|a_h\|_{0,\infty,\Omega} \le C\,\|a\|_{0,\infty,\Omega}.$$

The fitted finite element space is given by

$$W_h := V_h + X_h \; (\subset H_0^1(\Omega))$$
$$X_h := \{v \in C^0(\overline{\Omega});\; v_{|\Omega\setminus S_h^\gamma} = 0,\; v_{|K} \in P_1(K)\; \forall\, K \in \mathscr{T}_T^\gamma,\; \forall\, T \in \mathcal{T}_h^\gamma\}$$

Whence the Fitted Finite Element Method:

Find $u_h^F \in W_h$ such that $\displaystyle\int_\Omega a_h \nabla u_h^F \cdot \nabla v\,dx = \int_\Omega fv\,dx \qquad \forall\, v \in W_h.$

We assume (a weaker mesh regularity) that for some $\theta \in [0,1)$,

$$\frac{h}{\varrho_K} \le C\,h^{-\theta} \qquad \forall\, K \in \mathscr{T}_T^\gamma,\; T \in \mathscr{T}_h^\gamma.$$

where ϱ_K is the diameter of the inscribed circle in K.

Theorem

We have the error estimate

$$|u - u_h^F|_{1,\Omega} \leq \begin{cases} C\, h^{1-\theta}\, \|u\|_{2,\Omega^+\cup\Omega^-} & \text{if } u \in H^2(\Omega^+ \cup \Omega^-) \\ C\, h\, \|u\|_{2,\Omega^+\cup\Omega^-} & \text{if } u \in W^{2,\infty}(\Omega^+ \cup \Omega^-). \end{cases}$$

This method is rather simple but, in view of a time (or iteration) dependent interface, it implies a variable matrix structure.

To avoid this drawback, we resort to a hybrid technique.

Theorem

We have the error estimate

$$|u - u_h^F|_{1,\Omega} \leq \begin{cases} C\, h^{1-\theta}\, \|u\|_{2,\Omega^+\cup\Omega^-} & \textit{if } u \in H^2(\Omega^+ \cup \Omega^-) \\ C\, h\, \|u\|_{2,\Omega^+\cup\Omega^-} & \textit{if } u \in W^{2,\infty}(\Omega^+ \cup \Omega^-). \end{cases}$$

This method is rather simple but, in view of a time (or iteration) dependent interface, it implies a variable matrix structure.

To avoid this drawback, we resort to a hybrid technique.

Theorem

We have the error estimate

$$|u - u_h^F|_{1,\Omega} \le \begin{cases} C\, h^{1-\theta}\, \|u\|_{2,\Omega^+ \cup \Omega^-} & \text{if } u \in H^2(\Omega^+ \cup \Omega^-) \\ C\, h\, \|u\|_{2,\Omega^+ \cup \Omega^-} & \text{if } u \in W^{2,\infty}(\Omega^+ \cup \Omega^-). \end{cases}$$

This method is rather simple but, in view of a time (or iteration) dependent interface, it implies a variable matrix structure.

To avoid this drawback, we resort to a hybrid technique.

Theorem

We have the error estimate

$$|u - u_h^F|_{1,\Omega} \leq \begin{cases} C\, h^{1-\theta}\, \|u\|_{2,\Omega^+ \cup \Omega^-} & \text{if } u \in H^2(\Omega^+ \cup \Omega^-) \\ C\, h\, \|u\|_{2,\Omega^+ \cup \Omega^-} & \text{if } u \in W^{2,\infty}(\Omega^+ \cup \Omega^-). \end{cases}$$

This method is rather simple but, in view of a time (or iteration) dependent interface, it implies a variable matrix structure.

To avoid this drawback, we resort to a hybrid technique.

A hybrid formulation

As usual, we start by defining a pseudo-continuous hybrid method. Let

$$\widehat{Z}_h := H_0^1(\Omega) + \widehat{X}_h,$$
$$\widehat{X}_h := \{v \in L^2(\Omega);\ v_{|\Omega \setminus S_h^\gamma} = 0,\ v_{|T} \in H^1(T)\ \forall\ T \in \mathscr{T}_h^\gamma\},$$
$$\widehat{Q}_h := \prod_{e \in \mathscr{E}_h^\gamma} (H_{00}^{\frac{1}{2}}(e))',$$

where

$$H_{00}^{\frac{1}{2}}(e) := \{v_{|e};\ v \in H^1(T),\ e \in \mathscr{E}_T, v = 0 \text{ on } d\ \forall\ d \in \mathscr{E}_T, d \neq e\}.$$

We define the problem

Find $(\widehat{u}_h^H, \widehat{\lambda}_h) \in \widehat{Z}_h \times \widehat{Q}_h$ such that:

$$\sum_{T \in \mathscr{T}_h} \int_T a_h \nabla \widehat{u}_h^H \cdot \nabla v\, dx - \sum_{e \in \mathscr{E}_h^\gamma} \int_e \widehat{\lambda}_h [v]\, ds = \int_\Omega fv\, dx \qquad \forall\ v \in \widehat{Z}_h,$$

$$\sum_{e \in \mathscr{E}_h^\gamma} \int_e \mu\, [\widehat{u}_h^H]\, ds = 0 \qquad \forall\ \mu \in \widehat{Q}_h.$$

Theorem

The previous problem has a unique solution. Moreover

$$\widehat{u}_h^H \in H_0^1(\Omega), \quad \widehat{\lambda}_h = a_h \frac{\partial \widehat{u}_h^H}{\partial n}.$$

We define the spaces:

$$Z_h := V_h + Y_h,$$
$$V_h := \{v \in C^0(\overline{\Omega});\ v_{|T} \in P_1(T)\ \forall\ T \in \mathscr{T}_h\},$$
$$Y_h := \{v \in L^2(\Omega);\ v_{|\Omega \setminus S_h^\gamma} = 0,\ v_{|K} \in P_1(K)\ \forall\ K \in \mathscr{T}_T^\gamma,\ \forall\ T \in \mathscr{T}_h^\gamma\},$$
$$Q_h := \{\mu \in L^2(\prod_{e \in \mathscr{E}_h^\gamma});\ \mu_{|e} = \text{const.}\ \forall\ e \in \mathscr{E}_h^\gamma\}.$$

The discrete problem is:

Find $(u_h^H, \lambda_h) \in Z_h \times Q_h$ such that

$$\sum_{T \in \mathscr{T}_h} \int_T a_h \nabla u_h^H \cdot \nabla v\, dx - \sum_{e \in \mathscr{E}_h^\gamma} \int_e \lambda_h [v]\, ds = \int_\Omega f v\, dx \qquad \forall\ v \in Z_h,$$

$$\sum_{e \in \mathscr{E}_h^\gamma} \int_e \mu [u_h^H]\, ds = 0 \qquad \forall\ \mu \in Q_h.$$

A hybrid finite element method

We define the spaces:

$$Z_h := V_h + Y_h,$$
$$V_h := \{v \in C^0(\overline{\Omega});\ v_{|T} \in P_1(T)\ \forall\ T \in \mathscr{T}_h\},$$
$$Y_h := \{v \in L^2(\Omega);\ v_{|\Omega\setminus S_h^\gamma} = 0,\ v_{|K} \in P_1(K)\ \forall\ K \in \mathscr{T}_T^\gamma,\ \forall\ T \in \mathscr{T}_h^\gamma\},$$
$$Q_h := \{\mu \in L^2(\prod_{e\in\mathscr{E}_h^\gamma});\ \mu_{|e} = \text{const.}\ \forall\ e \in \mathscr{E}_h^\gamma\}.$$

The discrete problem is:

Find $(u_h^H, \lambda_h) \in Z_h \times Q_h$ such that

$$\sum_{T\in\mathscr{T}_h} \int_T a_h \nabla u_h^H \cdot \nabla v\, dx - \sum_{e\in\mathscr{E}_h^\gamma} \int_e \lambda_h [v]\, ds = \int_\Omega fv\, dx \qquad \forall\ v \in Z_h,$$

$$\sum_{e\in\mathscr{E}_h^\gamma} \int_e \mu [u_h^H]\, ds = 0 \qquad \forall\ \mu \in Q_h.$$

A hybrid finite element method (Cont'd)

Remark

The advantage of the hybrid approximation is that the added degrees of freedom can be locally eliminated (at element level).

Lemma

The hybrid approximation problem has a unique solution. In addition, we have

$$\|u_h^H\|_{\widehat{Z}_h} + \|\lambda_h\|_{Q_h} \leq C\,\|f\|_{0,\Omega}.$$

and

$$[u_h^H] = 0 \qquad \text{on } e,\ \forall\, e \in \mathscr{T}_h^\gamma.$$

A hybrid finite element method (Cont'd)

Remark

The advantage of the hybrid approximation is that the added degrees of freedom can be locally eliminated (at element level).

Lemma

The hybrid approximation problem has a unique solution. In addition, we have

$$\|u_h^H\|_{\widehat{Z}_h} + \|\lambda_h\|_{Q_h} \leq C\,\|f\|_{0,\Omega}.$$

and

$$[u_h^H] = 0 \qquad \text{on } e,\ \forall\ e \in \mathscr{T}_h^{\gamma}.$$

Finally, we have the convergence result:

Theorem

We have the error bound

$$\|u - u_h^H\|_{\widehat{Z}_h} \le \begin{cases} C\,h^{1-\theta}\,|f|_{0,\Omega} & \text{if } u \in H^2(\Omega^- \cup \Omega^+), \\ C\,h\,\|f\|_{0,\Omega} & \text{if } u \in W^{2,\infty}(\Omega^- \cup \Omega^+). \end{cases}$$

A numerical test

We consider the case of a radial solution in the square $\Omega = (-1,1)^2$, with

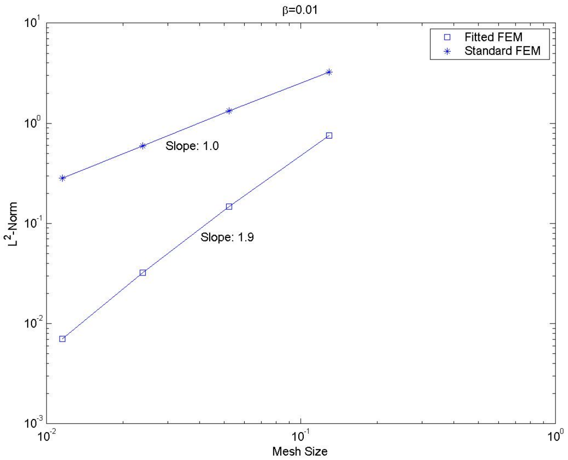
$$\Omega^- = \Big\{x \in \Omega;\ |x| < R\Big\},\ \Omega^+ = \Omega \setminus \Omega^-.$$

and

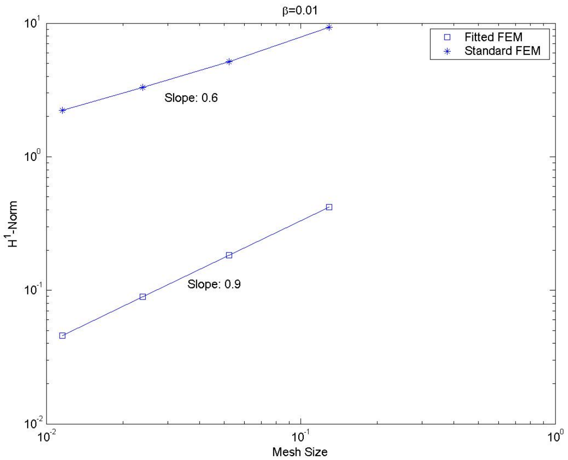
$$a = a^- \text{ in } \Omega^-, \quad a = a^+ \text{ in } \Omega^+, \quad \beta = \frac{a^+}{a^-}.$$

For $f = 1$, we have the solution

$$u(x) = \begin{cases} \frac{2-|x|^2}{4a^-} & \text{if } |x| < R \\ \frac{R^2-|x|^2}{4a^+} + \frac{2-R^2}{4a^-} & \text{if } |x| >= R \end{cases}$$

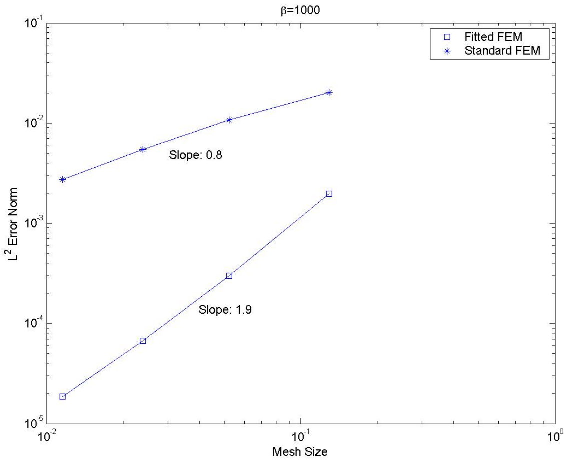


$\beta = 0.01$, Discrete L^2-norm



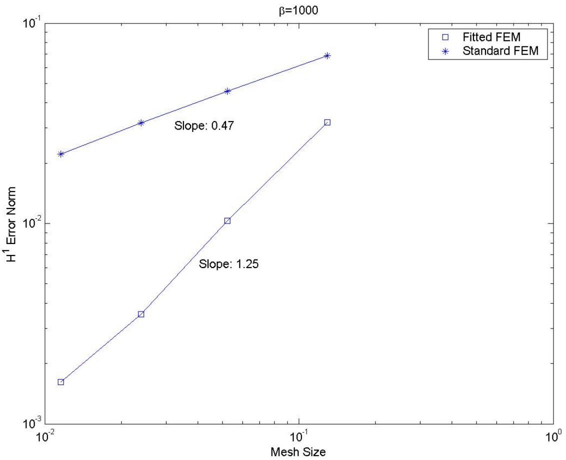
$\beta = 0.01$, Discrete H^1-norm

A numerical test (Cont'd)



$\beta = 1\,000$, Discrete L^2-norm

A numerical test (Cont'd)



$\beta = 1\,000$, Discrete H^1-norm