Derivation of mathematical models for eddy current systems with thin inductors

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joint work with Youcef Amirat

Circuit and field equations

- In modelling of electromagnetic processes, one can distinguish two main classes of models:
- Field models: Phenomena are described by partial differential equations in the whole space. These models are heavy to solve. They concern significant size conductors in which one is interested in field distribution (magnetic field, electric current, . . .)
- Circuit models: Processes are described by ordinary differential, integral or algebraic equations. Models are simpler to solve. They concern structures with complex connexions.
- Our purpose: To derive, (and mathematically justify), using asymptotic approximation, coupled models which use *field* equations in *thick* conductors and *circuit* equations in thin ones.
- Before treating Maxwell equations, we start here by considering *Eddy current* equations.
- Eddy current processes consist in generating electric current and magnetic field in conductors for the purpose of heating, stirring, metal shaping, welding, . . .

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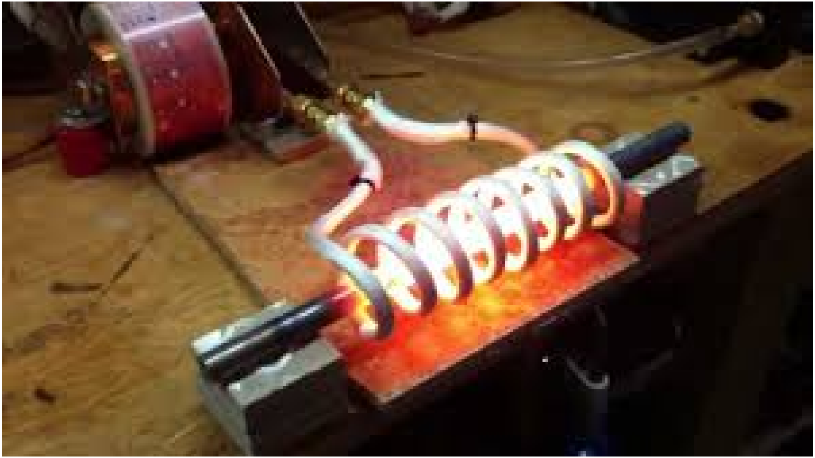

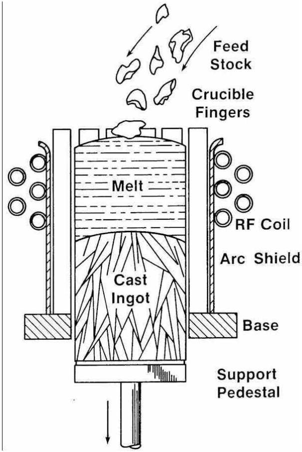
Feed
Stock
Crucible
Fingers
Melt
RF Coil
Arc Shield
Cast
Ingot
Base
Support
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- Electromagnetic eddy current setups are composed of conductors (workpieces) and inductors.
- Inductors are conductors that are connected to a power generator. The connection is idealized by a cut.
- Inductors may be thin wires or coils, whose thickness is smaller than other conductors.
- These setups are generally modelled by Maxwell's equations in which the wave propagation term is neglected (assuming low frequency). These equations are valid in the conductors and in the free space (vacuum or dielectric) domains.
- When the inductor is “thin”, an intuitive modelling consists in using a model where thick conductors are handled by a PDE, and coils (inductors) are handled by a (ODE or algebraic) circuit equation and interface conditions that couple all equations.

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In time harmonic regime ($\boldsymbol{U}(\boldsymbol{x},t) = \Re(e^{i\omega t}\boldsymbol{u}(\boldsymbol{x}))$), we have the set of equations:

$$
\begin{aligned}
& i\omega\varepsilon \boldsymbol{e} - \textbf{curl}\, \boldsymbol{h} + \boldsymbol{j} = 0, \\
& i\omega\mu \boldsymbol{h} + \textbf{curl}\, \boldsymbol{e} = 0, \\
& \varrho \boldsymbol{j} = \boldsymbol{e} && \text{in the conductors} \\
& \boldsymbol{j} = \boldsymbol{0} && \text{in the vacuum}
\end{aligned}
$$

where:

$\boldsymbol{h}$: Magnetic field, $\boldsymbol{e}$: Electric field, $\boldsymbol{j}$: Electric current

and

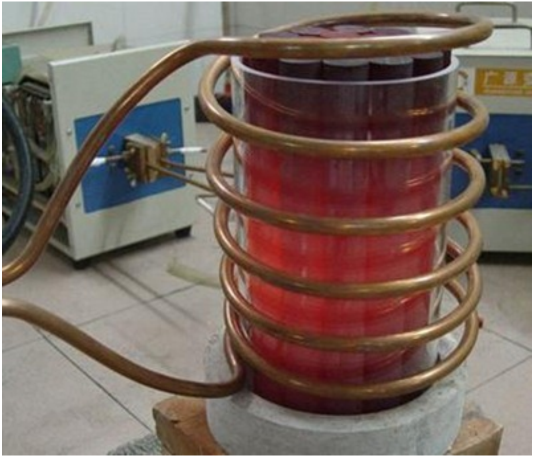
ω : Angular frequency,

ε : Electric permittivity,

μ : Magnetic permeability,

ϱ : Electric resistivity.

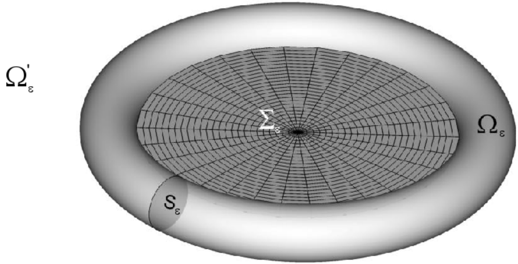
An example



The eddy current model

Let us consider a simple configuration with one inductor.
The various involved domains are:

- $\Omega = \Omega_\varepsilon$ is the inductor (a toroidal domain with thickness $\varepsilon \ll 1$), with boundary Γ_ε
- $S = S_\varepsilon$ is a cut in the inductor (s.t. $\dot{\Omega}_\varepsilon := \Omega_\varepsilon \setminus S_\varepsilon$ is simply connected)
- $\Omega'_\varepsilon : \mathbb{R}^3 \setminus \overline{\Omega}_\varepsilon$ is the free space
- Σ_ε is a cut in the free space (s.t. $\dot{\Omega}'_\varepsilon := \Omega'_\varepsilon \setminus \Sigma_\varepsilon$ is simply connected)



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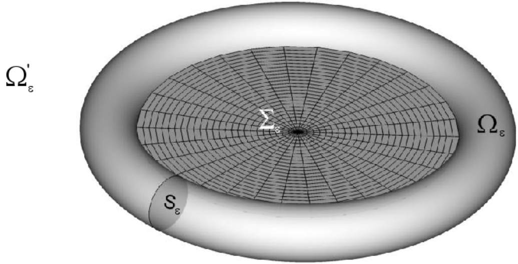

A well known result in vector analysis can be stated as follows:

Let $\boldsymbol{k} \in \boldsymbol{L}^2(\mathbb{R}^3)$ such that

$$\mathbf{curl}\,\boldsymbol{k} = 0 \qquad \text{in } \Omega_\varepsilon.$$

Then, there exists $\varphi \in H^1(\Omega_\varepsilon)$ and $\beta \in \mathbb{C}$ such that

$$\boldsymbol{k} = \nabla\varphi + \beta\,\nabla p^\varepsilon \qquad \text{in } \Omega_\varepsilon,$$

where p^ε is the unique solution of the problem:

$$\begin{cases} \Delta p^\varepsilon = 0 & \text{in } \dot{\Omega}_\varepsilon, \\ \dfrac{\partial p^\varepsilon}{\partial n} = 0 & \text{on } \Gamma_\varepsilon, \\ [p^\varepsilon]_{S_\varepsilon} = 1, \quad \left[\dfrac{\partial p^\varepsilon}{\partial n}\right]_{S_\varepsilon} = 0 & \end{cases}$$

We deduce formally that we have

$$\beta = \int_{S_\varepsilon} \mathbf{curl}\,\boldsymbol{k}\cdot\boldsymbol{n}\,ds = \int_{\partial S_\varepsilon} \boldsymbol{k}\cdot\boldsymbol{t}\,d\ell.$$

We have a similar result in the free space Ω'_ε.

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The equations

Main issue:
We consider eddy current equations with given voltage:

$$\begin{aligned} \mathrm{i}\omega\mu\boldsymbol{h}^\varepsilon + \mathbf{curl}\,\boldsymbol{e}^\varepsilon &= v\boldsymbol{\delta}_\varepsilon && \text{in } \Omega_\varepsilon, \\ \varrho^\varepsilon\,\mathbf{curl}\,\boldsymbol{h}^\varepsilon &= \boldsymbol{e}^\varepsilon && \text{in } \Omega_\varepsilon, \\ \mathbf{curl}\,\boldsymbol{h}^\varepsilon &= 0 && \text{in } \Omega'_\varepsilon \end{aligned}$$

where $\boldsymbol{\delta}_\varepsilon$ is the distribution (supported by the cut S_ε) given by

$$\langle \boldsymbol{\delta}_\varepsilon, \boldsymbol{w} \rangle := \int_{S_\varepsilon} \mathbf{curl}\,\boldsymbol{w} \cdot \boldsymbol{n}\, ds \qquad \forall\, \boldsymbol{w} \in \mathcal{C}_0^\infty(\mathbb{R}^3)^3.$$

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Consider the space

$$\boldsymbol{\mathcal{H}}_\varepsilon = \{\boldsymbol{k} \in \boldsymbol{L}^2(\mathbb{R}^3); \textbf{curl}\, \boldsymbol{k} \in \boldsymbol{L}^2(\mathbb{R}^3),\ \textbf{curl}\, \boldsymbol{k} = 0 \text{ in } \Omega'_\varepsilon\}$$

The time-harmonic variational problem reads:

$$\begin{cases} \boldsymbol{h}^\varepsilon \in \boldsymbol{\mathcal{H}}_\varepsilon, \\ \mathrm{i}\omega \int_{\mathbb{R}^3} \mu\, \boldsymbol{h}^\varepsilon \cdot \overline{\boldsymbol{k}}\, d\boldsymbol{x} + \int_{\Omega_\varepsilon} \varrho^\varepsilon\, \textbf{curl}\, \boldsymbol{h}^\varepsilon \cdot \textbf{curl}\, \overline{\boldsymbol{k}}\, d\boldsymbol{x} = v\langle \textbf{curl}\, \boldsymbol{k} \cdot \boldsymbol{n}, 1\rangle_{S_\varepsilon} \qquad \forall\ \boldsymbol{k} \in \boldsymbol{\mathcal{H}}_\varepsilon, \end{cases}$$

where $\langle \cdot, \cdot \rangle_{S_\varepsilon}$ is the duality pairing between $H^{\frac{1}{2}}(S_\varepsilon)'$ and $H^{\frac{1}{2}}(S_\varepsilon)$.

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&0 < \mu_0 \leq \mu \leq \mu_M && \text{a.e. in } \mathbb{R}^3, \\
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Lemma

(Amrouche et al., Math. Meth. Appl. Sci, 1998)
We have

$$\langle \operatorname{\mathbf{curl}} \boldsymbol{k} \cdot \boldsymbol{n}, 1 \rangle_{S_\varepsilon} = \int_{\dot{\Omega}_\varepsilon} \nabla p^\varepsilon \cdot \operatorname{\mathbf{curl}} \overline{\boldsymbol{k}} \, d\boldsymbol{x},$$

where p^ε is the potential defined previously.
Consequently, the total current running in the inductor is given by

$$I^\varepsilon := \langle \operatorname{\mathbf{curl}} \boldsymbol{h}^\varepsilon \cdot \boldsymbol{n}, 1 \rangle_{S_\varepsilon} = \int_{\dot{\Omega}_\varepsilon} \nabla p^\varepsilon \cdot \operatorname{\mathbf{curl}} \overline{\boldsymbol{h}}^\varepsilon \, d\boldsymbol{x}.$$

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Interpretation of v

Setting $\boldsymbol{k} = \boldsymbol{h}^\varepsilon$ in the variational formulation, we get

$$\mathrm{i}\omega \int_{\mathbb{R}^3} \mu |\boldsymbol{h}^\varepsilon|^2 \, d\boldsymbol{x} + \int_{\Omega_\varepsilon} \varrho^\varepsilon |\,\mathbf{curl}\,\boldsymbol{h}^\varepsilon|^2 \, d\boldsymbol{x} = v \langle \mathbf{curl}\,\boldsymbol{h}^\varepsilon \cdot \boldsymbol{n}, 1 \rangle_{S_\varepsilon}.$$

Let us define the electric current $\boldsymbol{j}^\varepsilon = \mathbf{curl}\,\boldsymbol{h}^\varepsilon$ and the electric field $\boldsymbol{e}^\varepsilon = \varrho^\varepsilon \boldsymbol{j}^\varepsilon$. We deduce:

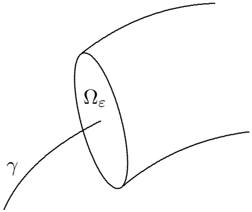
$$\mathrm{i}\omega \int_{\mathbb{R}^3} \mu \boldsymbol{h}^\varepsilon \cdot \overline{\boldsymbol{h}}^\varepsilon \, d\boldsymbol{x} + \int_{\Omega_\varepsilon} \boldsymbol{j}^\varepsilon \cdot \overline{\boldsymbol{e}}^\varepsilon \, d\boldsymbol{x} = v \langle \boldsymbol{j}^\varepsilon \cdot \boldsymbol{n}, 1 \rangle_{S_\varepsilon} =: v\, I^\varepsilon.$$

Magnetic Energy $+$ Electric Energy $=$ Voltage $\times$ Current

How does Ω_ε depend on ε ?

We consider a closed Jordan curve γ of class C^3, parameterized by a function $\boldsymbol{g} : (0,1) \to \mathbb{R}^3$ such that

$$\boldsymbol{g}(0) = \boldsymbol{g}(1),\ \ \boldsymbol{g}'(0) = \boldsymbol{g}'(1),\ \ |\boldsymbol{g}'(s)| \geq \alpha > 0.$$



We define the mapping

$$\boldsymbol{F}_\varepsilon(s,\xi,\theta) = \boldsymbol{g}(s) + \varepsilon r \xi(\cos\theta\, \boldsymbol{\nu}(s) + \sin\theta\, \boldsymbol{b}(s))$$

for $(s,\xi,\theta) \in [0,1) \times (0,1) \times [0,2\pi)$, where $\boldsymbol{\nu}$ is the principal normal and $\boldsymbol{b}$ is the bi-normal of the curve γ.

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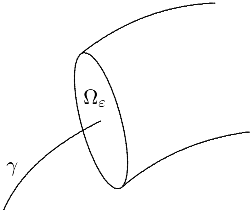

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The Jacobian of $\boldsymbol{F}_\varepsilon$ is given by

$$J_\varepsilon(s,\xi,\theta) = \varepsilon^2 r^2 \xi(|\boldsymbol{g}'(s)| - \varepsilon r \xi \kappa(s)) \qquad \kappa : \text{Curvature}$$

It is easy to see that if we have the property

$$r|\kappa(s)| \leq \alpha \qquad s \in [0,1),$$

then $\boldsymbol{F}_\varepsilon$ is a C^1–diffeomorphism of $\widehat{\Omega}$ onto Ω_ε that defines $\Omega_\varepsilon := \boldsymbol{F}_\varepsilon(\widehat{\Omega})$.

Lemma

We have the asymptotic behaviour

$$\int_{\dot{\Omega}_\varepsilon} |\nabla p^\varepsilon|^2 \, d\boldsymbol{x} = \frac{\pi r^2}{\ell} \varepsilon^2 + O(\varepsilon^3).$$

Theorem

There is a constant $C > 0$, independent of ε such that:

$$\|\boldsymbol{h}^\varepsilon\|_{L^2(\mathbb{R}^3)} \le C,$$
$$\|\,\mathbf{curl}\,\boldsymbol{h}^\varepsilon\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon^{-1},$$
$$|I^\varepsilon| \le C.$$

where we recall that

$$I^\varepsilon := \langle \mathbf{curl}\,\boldsymbol{h}^\varepsilon \cdot \boldsymbol{n}, 1\rangle_{S_\varepsilon} = \int_{\dot{\Omega}_\varepsilon} \nabla p^\varepsilon \cdot \mathbf{curl}\,\boldsymbol{h}\, dx.$$

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$$\|\operatorname{\mathbf{curl}} \boldsymbol{h}^\varepsilon\|_{\boldsymbol{L}^2(\Omega_\varepsilon)} \le C\varepsilon^{-1},$$
$$|I^\varepsilon| \le C.$$

where we recall that

$$I^\varepsilon := \langle \operatorname{\mathbf{curl}} \boldsymbol{h}^\varepsilon \cdot \boldsymbol{n}, 1 \rangle_{S_\varepsilon} = \int_{\dot{\Omega}_\varepsilon} \nabla p^\varepsilon \cdot \operatorname{\mathbf{curl}} \boldsymbol{h} \, d\boldsymbol{x}.$$

Theorem (*Arch. Rat. Mech. Anal., 2017, and submitted paper*)

We have the asymptotic approximation:

$$\Big(\frac{\mathrm{i}\omega}{|\ln\varepsilon|}L^\varepsilon + R\Big)I^\varepsilon = v + \mathcal{O}(|\ln\varepsilon|^{-\frac{1}{2}})$$

where R *is the conductor resistance and* L^ε *is its inductance,* i.e.

$$R = \frac{\varrho^\varepsilon \ell}{\pi\varepsilon^2 r^2} = \frac{\varrho\ell}{\pi r^2}, \qquad L^\varepsilon = \mu_0 \int_{\dot\Omega'_\varepsilon} |\nabla q^\varepsilon|^2\,d\boldsymbol{x},$$

ℓ *is the length of* γ *and* q^ε *is a solution of the problem:*

$$\begin{cases} \Delta q^\varepsilon = 0 & \text{in } \dot\Omega'_\varepsilon, \\ \dfrac{\partial q^\varepsilon}{\partial n} = 0 & \text{on } \Gamma_\varepsilon, \\ [q^\varepsilon]_{\Sigma_\varepsilon} = 1,\ \Big[\dfrac{\partial q^\varepsilon}{\partial n}\Big]_{\Sigma_\varepsilon} = 0 \end{cases}$$

In addition, the limit current $I := \lim_{\varepsilon\to 0} I^\varepsilon$ *satisfies the Kirchhoff's circuit equation:*

$$\Big(\frac{\mathrm{i}\omega\mu_0\ell}{2\pi} + \frac{\varrho\ell}{\pi r^2}\Big)I = v.$$

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has a unique solution.

We have proven in a previous work *(M^{3}AS, 2002)*, the asymptotic behaviour:

$$L^\varepsilon = \mu_0 \int_{\dot{\Omega}'_\varepsilon} |\nabla q^\varepsilon|^2 \, d\boldsymbol{x} = \frac{\mu_0 \ell}{2\pi} |\ln \varepsilon| + \mathcal{O}(1).$$

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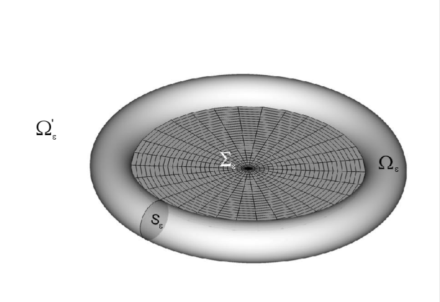
Ω'_ε
Σ_ε
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A sketch of the proof

Key idea: Choice of a convenient test function.

Let us take $\boldsymbol{k} = \boldsymbol{k}^\varepsilon$ in the variational formulation:

$$\mathrm{i}\omega \int_{\mathbb{R}^3} \mu\, \boldsymbol{h}^\varepsilon \cdot \overline{\boldsymbol{k}}^\varepsilon \, d\boldsymbol{x} + \varepsilon^2 \int_{\Omega_\varepsilon} \varrho\, \mathbf{curl}\, \boldsymbol{h}^\varepsilon \cdot \mathbf{curl}\, \overline{\boldsymbol{k}}^\varepsilon \, d\boldsymbol{x} = v \langle \mathbf{curl}\, \boldsymbol{k}^\varepsilon \cdot \boldsymbol{n}, 1 \rangle_{S_\varepsilon}.$$

Since $\mathbf{curl}\, \boldsymbol{h}^\varepsilon = 0$ in Ω'_ε, we have the expansion

$$\boldsymbol{h}^\varepsilon = \nabla \varphi^\varepsilon + I^\varepsilon \nabla \boldsymbol{q}^\varepsilon \quad \text{in } \dot{\Omega}'_\varepsilon.$$

Therefore

$$\mathrm{i}\omega \int_{\Omega_\varepsilon} \mu\, \boldsymbol{h}^\varepsilon \cdot \overline{\boldsymbol{k}}^\varepsilon \, d\boldsymbol{x}$$

$$+$$

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A sketch of the proof (Cont'd)

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Hint.

- If we choose $\boldsymbol{k}^\varepsilon = \alpha \nabla \boldsymbol{q}^\varepsilon$ in $\dot\Omega'_\varepsilon$, we have, since $\Delta \boldsymbol{q}^\varepsilon = 0$ in $\dot\Omega'_\varepsilon$:

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Let us then choose

$$\boldsymbol{k}^\varepsilon = \begin{cases} \dfrac{\varepsilon}{|\ln \varepsilon|}\dfrac{\pi r^2}{\ell}\nabla\widetilde{q}^\varepsilon & \text{in } \dot{\Omega}'_\varepsilon, \\ \textbf{curl}\, \boldsymbol{a}^\varepsilon & \text{in } \Omega_\varepsilon, \end{cases}$$

where $\widetilde{q}^\varepsilon$ is a regularization of q^ε (Note that $\nabla q^\varepsilon \in L^\alpha_{\text{loc}}(\Omega'_\varepsilon)$ for $1 \le \alpha < 2$), and

$$\begin{cases} \textbf{curl}\,\textbf{curl}\, \boldsymbol{a}^\varepsilon = \dfrac{1}{\varepsilon}\nabla p^\varepsilon & \text{in } \Omega_\varepsilon, \\ \operatorname{div} \boldsymbol{a}^\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ \boldsymbol{a}^\varepsilon \cdot \boldsymbol{n} = 0 & \text{on } \Gamma_\varepsilon, \\ \textbf{curl}\, \boldsymbol{a}^\varepsilon \times \boldsymbol{n} = \boldsymbol{k}^\varepsilon_{|\Omega'_\varepsilon} \times \boldsymbol{n} & \text{on } \Gamma_\varepsilon, \\ \langle\, \boldsymbol{a}^\varepsilon \cdot \boldsymbol{n}, 1\,\rangle_{S_\varepsilon} = 0. \end{cases}$$

A sketch of the proof (Cont'd)

Using this test function, we obtain

$$\mathrm{i}\omega\int_{\Omega_\varepsilon}\mu\boldsymbol{h}^\varepsilon\cdot\mathbf{curl}\,\boldsymbol{a}^\varepsilon\,d\boldsymbol{x}+\frac{\varepsilon}{|\ln\varepsilon|}\frac{\mathrm{i}\omega\mu_0\pi r^2}{\ell}\int_{\dot\Omega'_\varepsilon}\boldsymbol{h}^\varepsilon\cdot\nabla\widetilde{q}^\varepsilon\,d\boldsymbol{x}+\varepsilon I^\varepsilon=\frac{v}{\varepsilon}\int_{\dot\Omega_\varepsilon}|\nabla p^\varepsilon|^2\,d\boldsymbol{x}$$

Using the expansion $\boldsymbol{h}^\varepsilon=\nabla\varphi^\varepsilon+I^\varepsilon\nabla q^\varepsilon$ in $\dot\Omega'_\varepsilon$, we obtain

$$\Big(\frac{\mathrm{i}\omega}{|\ln\varepsilon|}+\frac{\varrho\ell}{\pi r^2}\Big)I^\varepsilon=\varepsilon^{-2}\frac{v\ell}{\pi r^2}\int_{\dot\Omega_\varepsilon}|\nabla p^\varepsilon|^2\,d\boldsymbol{x}-R_1^\varepsilon-R_2^\varepsilon$$

where

$$R_1^\varepsilon=\frac{\mathrm{i}\omega\ell}{\varepsilon\pi r^2}\int_{\Omega_\varepsilon}\mu\boldsymbol{h}^\varepsilon\cdot\mathbf{curl}\,\boldsymbol{a}^\varepsilon\,d\boldsymbol{x},$$

$$R_2^\varepsilon=\frac{\mathrm{i}\omega\mu_0}{|\ln\varepsilon|}\int_{\dot\Omega'_\varepsilon}\boldsymbol{h}^\varepsilon\cdot(\nabla\widetilde{q}^\varepsilon-\nabla q^\varepsilon)\,d\boldsymbol{x}.$$

We aim at proving that

$$\lim_{\varepsilon\to0}R_1^\varepsilon=\lim_{\varepsilon\to0}R_2^\varepsilon=0.$$

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where

$$R_1^\varepsilon = \frac{\mathrm{i}\omega\ell}{\varepsilon\pi r^2} \int_{\Omega_\varepsilon} \mu \boldsymbol{h}^\varepsilon \cdot \mathbf{curl}\, \boldsymbol{a}^\varepsilon \, d\boldsymbol{x},$$

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We aim at proving that

$$\lim_{\varepsilon \to 0} R_1^\varepsilon = \lim_{\varepsilon \to 0} R_2^\varepsilon = 0.$$

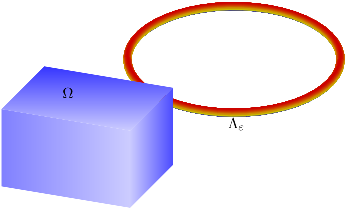
We prove the bounds:

$$|R_1^\varepsilon| \le C\varepsilon|\ln\varepsilon|^{-1},$$
$$|R_2^\varepsilon| \le C\varepsilon|\ln\varepsilon|^{-\frac{1}{2}}.$$

Then

$$\Big(\frac{1}{|\ln\varepsilon|}\mathrm{i}\omega L^\varepsilon + R^\varepsilon\Big)I^\varepsilon = v + \mathcal{O}(|\ln\varepsilon|^{-\frac{1}{2}})$$

We now consider the case of 2 domains Ω and Λ_ε such that $d(\overline{\Omega},\overline{\Lambda}_\varepsilon) \geq \delta > 0$:



We define the domains:

$$\mathcal{V}_\varepsilon = \mathbb{R}^3 \setminus (\overline{\Omega} \cup \overline{\Lambda}_\varepsilon), \quad \dot{\mathcal{V}}_\varepsilon = \mathcal{V}_\varepsilon \setminus \Sigma_\varepsilon,$$

and the space

$$\mathcal{H}_\varepsilon = \{\boldsymbol{k} \in \boldsymbol{L}^2(\mathbb{R}^3);\ \textbf{curl}\,\boldsymbol{k} \in \boldsymbol{L}^2(\mathbb{R}^3),\ \textbf{curl}\,\boldsymbol{k} = 0 \text{ in } \mathcal{V}_\varepsilon\}.$$

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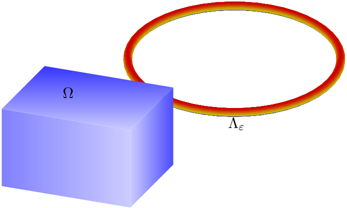

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We consider the variational formulation:

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Using the expansion of functions $\boldsymbol{k} \in \boldsymbol{\mathcal{H}}_\varepsilon$:

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where

$$\boldsymbol{\mathcal{X}}_\varepsilon = \big\{(\boldsymbol{k}, \psi, \beta) \in \boldsymbol{L}^2(\Omega \cup \Lambda_\varepsilon) \times W^1(\mathcal{V}_\varepsilon) \times \mathbb{C};$$
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Using the same technique as for the unique conductor case, we have the estimates:

$$\|\boldsymbol{h}^\varepsilon\|_{\boldsymbol{L}^2(\mathbb{R}^3)} + \varepsilon\, \|\, \mathbf{curl}\, \boldsymbol{h}^\varepsilon\|_{\boldsymbol{L}^2(\Lambda_\varepsilon)} + \|\, \mathbf{curl}\, \boldsymbol{h}^\varepsilon\|_{\boldsymbol{L}^2(\Omega)} + |I^\varepsilon| \le C.$$

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Define the space

$$\boldsymbol{\mathcal{Y}}_\varepsilon = \Big\{(\boldsymbol{k}, \psi, \beta) \in \boldsymbol{L}^2(\mathbb{R}^3) \times W^1(\Omega') \times \mathbb{C};\ \boldsymbol{k} \times \boldsymbol{n} = \nabla\psi \times \boldsymbol{n} + \frac{1}{|\ln \varepsilon|}\beta \nabla \boldsymbol{q}^\varepsilon \times \boldsymbol{n} \text{ on } \partial\Omega\Big\}$$

we obtain the approximate problem:

$$\begin{cases} (\boldsymbol{h}^\varepsilon, \varphi^\varepsilon, I^\varepsilon) \in \boldsymbol{\mathcal{Y}}_\varepsilon, \\ \mathrm{i}\omega \int_\Omega \mu \boldsymbol{h}^\varepsilon \cdot \overline{\boldsymbol{k}}\, d\boldsymbol{x} + \mathrm{i}\omega \int_{\Omega'} \nabla\varphi^\varepsilon \cdot \nabla\overline{\psi}\, d\boldsymbol{x} + \int_\Omega \varrho\, \mathbf{curl}\, \boldsymbol{h}^\varepsilon \cdot \mathbf{curl}\, \overline{\boldsymbol{k}}\, d\boldsymbol{x} \\ \qquad + \Big(\frac{\mathrm{i}\omega}{|\ln \varepsilon|} L^\varepsilon + R^\varepsilon\Big) I^\varepsilon \overline{\beta} = v\overline{\beta} \qquad \forall\, (\boldsymbol{k}, \psi, \beta) \in \boldsymbol{\mathcal{Y}}_\varepsilon \end{cases}$$